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Current fluctuations in the exclusion process and Bethe ansatz

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Abstract
We use the Bethe ansatz to derive analytical expressions for the current statistics in the asymmetric exclusion process with both forward and backward jumps. The Bethe equations are highly coupled and this fact has impeded their use in deriving exact results for finite systems. We overcome this technical difficulty by reformulating Bethe equations into a one-variable polynomial problem, akin to the functional Bethe ansatz. The perturbative solution of this equation leads to the cumulants of the current. We calculate here the first two orders and derive exact formulae for the mean value of the current and its fluctuations.

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1. Introduction

The asymmetric exclusion process (ASEP) plays a seminal role in non-equilibrium physics of low dimensional systems [1]. In its simplest version, the ASEP describes a system of particles, randomly hopping on a lattice with the hard-core exclusion interaction so that a lattice site can be occupied by only one particle at a given time. Due to its minimal character, this model appears as a building block in many seemingly unrelated fields [2]. By virtue of different mappings, the ASEP can be interpreted as a model for RNA transcription [3], hopping conductivity, polymers in random media, surface growth [4], traffic flow, molecular motors [5] etc. In the one-dimensional case, many exact results have been derived for the ASEP (for a review, see, e.g. [6, 7]). As a result, the relations between the intrinsic stochasticity of the dynamics, the external drive and the particle interactions are better understood. The fact that ASEP in one-dimension is an exactly solvable model should not be considered as just an elegant mathematical anomaly at odds with physical relevance. Indeed, many of the exact results obtained for ASEP have shed light on the behaviour of general driven diffusive systems by providing us with effective phenomenological descriptions that can be applied to more realistic models [8]. Examples of such descriptions that stem from mathematical results are: shock fronts to model boundary induced phase transitions, the interpretation of shocks as real space condensation (related to zero range processes) [9, 10], and the additivity principle [11]. Besides, the ASEP is a good toy-model to test the validity of general claims about non-equilibrium systems: for example, the Gallavotti–Cohen fluctuation theorem is satisfied by the ASEP (and by more general Markovian systems) as can be shown by elementary methods [12] whereas the proof for deterministic dynamical systems requires some restrictive hypothesis and is far more technical.

Exact solutions for the exclusion process have been obtained by using several different approaches, and in particular the matrix product representation and the Bethe ansatz. The matrix product representation was first introduced in [13] to study the stationary state and the phase diagram of the ASEP with open boundaries. The main idea comprises representing the stationary state as a trace over a suitable, usually quadratic, algebra; this technique has been generalized to many different models, including systems with shock profiles and with different classes of particles [14–16, 18]. An exhaustive and pedagogical review on the matrix method can be found in [19]. The Bethe ansatz was first used to calculate the spectral gap of the ASEP and the associated dynamical exponent [20–24]. Indeed, the Markov matrix of the ASEP that encodes its stochastic evolution can be mapped exactly onto a non-Hermitian spin chain Hamiltonian which is integrable. The Bethe ansatz also allows us to study spectral degeneracies [25], and to investigate variants of the ASEP and more general particle hopping processes [26–29] (for a review see, e.g. [30]).

A particularly important physical quantity in the ASEP is the statistics of the current in the stationary regime. This current becomes a local height variable when the ASEP is translated into a random solid on solid model that describes the growth of a random interface. Indeed, in this mapping, a forward random jump of a particle through a bond corresponds to a random deposition event of a unit ‘brick’ on the interface; a backward jump corresponds to the evaporation of a brick. The time integrated current through a bond of the ASEP is therefore equivalent to the total height of the interface at a given point. In the continuous limit, the motion of this interface is described by the Kardar–Parisi–Zhang (KPZ) equation (see, e.g. [4]). The exclusion process in one dimension is thus a discretized version of the KPZ equation and exact results about the ASEP have therefore interesting interpretations in terms of surface growth.

For the exclusion process on a periodic ring, the mean value of the stationary current through a bond can be easily derived from elementary combinatorics; in the limit of a large system the mean current is given by the density of vacancies multiplied by the asymmetry rate. However, the higher moments of the current in the stationary state are much more difficult to calculate. In fact, the full statistics of the current was determined only for the particular case of the *totally* asymmetric exclusion process (TASEP), where the particles are allowed to jump only in one direction. For any system size, an analytical expression for the cumulant generating function was obtained, leading to an exact formula for the large deviation function [31, 32]. This result was derived using the Bethe equations which, for the TASEP, can be solved explicitely thanks to a decoupling property that reduces them to a one-variable polynomial equation plus a self-consistency condition [21, 27, 30].

In the general case, when jumps on both directions are allowed, the Bethe equations do not decouple and it has not been possible to use them to derive exact results for finite systems. An exact formula for the fluctuation of the current (i.e. the second moment of the current) in the long time limit could however be derived using an extension of the matrix method [33, 34]. But higher moments appeared to be out of reach.

The aim of the present work is to derive analytical results for the current statistics in ASEP with forward and backward jumps (sometimes called the partially asymmetric exclusion process) from the Bethe ansatz. We overcome the technical difficulty that hindered the solution of the Bethe equations in the general case by reducing them to an effective one-variable problem thanks to a suitable reformulation, akin to the so-called functional Bethe ansatz. This one-variable equation can be interpreted as a purely algebraic problem involving a divisibility condition between two polynomials. In this work, we use this formalism to derive the expressions of the mean value of the current and its variance. Our technique can be used to calculate the current cumulant to any desired order.

The outline of this work is as follows. In section 2, we explain that the cumulant generating function can be expressed as the maximal eigenvalue of a suitable deformation of the Markov matrix where the deformation parameter represents the fugacity of the jumps. In section 3, we give the Bethe equations that allow us to diagonalize this matrix. The reformulation of the Bethe equations as a problem in polynomial divisibility is done in section 4. In section 5, we solve perturbatively this purely algebraic problem to the second order with respect to the jump fugacity. This allows us to derive the exact formulae for the mean value and the variance of the current in section 6. The last section is devoted to concluding remarks. Some technical derivations are given in the appendices.

2. Current statistics as an eigenvalue problem

2.1. The asymmetric exclusion process

The exclusion process on a periodic one-dimensional lattice with L sites (sites i and $L+i$ are identical) is a stochastic interacting particle model in which each lattice site is occupied by at most one particle at a given time (*exclusion rule*). The system evolves with time according to a stochastic dynamics: a particle on a site i at time t jumps, in the interval between t and $t+\mathrm{d}t$, with probability $p\,\mathrm{d}t$ to the neighbouring site $i+1$ if this site is empty and with probability $q\,\mathrm{d}t$ to the site $i-1$ if this site is empty. The jump rates p and q are normalized such that $p+q=1$. The special case where the jumps are totally biased in one direction ($p=1$ or $q=1$) is called the totally asymmetric exclusion process (TASEP). For $p=q=1/2$, the exclusion process is symmetric (SEP). If the number of particles in the ring is N, the total number of configurations is given by the binomial coefficient $\binom{L}{N}$.

We call $P_t(\mathcal{C})$ the probability that the system is in the configuration $\mathcal{C}$ at time t. As the exclusion process is a continuous-time Markov process, the time evolution of $P_t(\mathcal{C})$ is determined by the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}P_t(\mathcal{C}) = \sum_{\mathcal{C}'} M(\mathcal{C},\mathcal{C}')P_t(\mathcal{C}') = \sum_{\mathcal{C}'}(M_0(\mathcal{C},\mathcal{C}')+M_1(\mathcal{C},\mathcal{C}')+M_{-1}(\mathcal{C},\mathcal{C}'))P_t(\mathcal{C}'). \qquad (1)$$

The Markov matrix M encodes the dynamics of the exclusion process: the non-diagonal element $M_1(\mathcal{C},\mathcal{C}')$ represents the transition rate from configuration $\mathcal{C}'$ to $\mathcal{C}$ where a particle hops in the forward (i.e. anti-clockwise) direction, the non-diagonal element $M_{-1}(\mathcal{C},\mathcal{C}')$ represents the transition rate from configuration $\mathcal{C}'$ to $\mathcal{C}$ where a particle hops in the backward (i.e. clockwise) direction. The diagonal term $M_0(\mathcal{C},\mathcal{C}) = -\sum_{\mathcal{C}'\neq\mathcal{C}}(M_1(\mathcal{C}',\mathcal{C})+M_{-1}(\mathcal{C}',\mathcal{C}))$ represents the exit rate from the configuration $\mathcal{C}$.

2.2. Generalized master equation for current statistics

We call Y_t the total distance covered by all the particles between time 0 and time t and $P_t(\mathcal{C},Y)$ the joint probability of being in the configuration $\mathcal{C}$ at time t with $Y_t = Y$. An evolution equation, analogous to equation (1), can be written for $P_t(\mathcal{C}, Y)$ as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} P_t(\mathcal{C}, Y) = \sum_{\mathcal{C}'} (M_0(\mathcal{C}, \mathcal{C}') P_t(\mathcal{C}', Y) + M_1(\mathcal{C}, \mathcal{C}') P_t(\mathcal{C}', Y-1) + M_{-1}(\mathcal{C}, \mathcal{C}') P_t(\mathcal{C}', Y+1)). \tag{2}$$

We now recall how the full statistics of Y_t can be determined [12, 31]. In terms of the generating function $F_t(\mathcal{C})$ defined as

$$F_t(\mathcal{C}) = \sum_{Y=-\infty}^{+\infty} \mathrm{e}^{\gamma Y} P_t(\mathcal{C}, Y), \tag{3}$$

equation (2) takes the simpler form:

$$\frac{\mathrm{d}}{\mathrm{d}t} F_t(\mathcal{C}) = \sum_{\mathcal{C}'} (M_0(\mathcal{C}, \mathcal{C}') + \mathrm{e}^{\gamma} M_1(\mathcal{C}, \mathcal{C}') + \mathrm{e}^{-\gamma} M_{-1}(\mathcal{C}, \mathcal{C}')) F_t(\mathcal{C}') = \sum_{\mathcal{C}'} M(\gamma)(\mathcal{C}, \mathcal{C}') F_t(\mathcal{C}'). \tag{4}$$

This equation is similar to the original Markov equation (1) for the probability distribution $P_t(\mathcal{C})$ but where the original Markov matrix M is deformed into $M(\gamma)$ which is given by

$$M(\gamma) = M_0 + \mathrm{e}^{\gamma} M_1 + \mathrm{e}^{-\gamma} M_{-1}. \tag{5}$$

We emphasize that $M(\gamma)$, that governs the evolution of $F_t(\mathcal{C})$, is not a Markov matrix for $\gamma \neq 0$ (the sum of the elements in a given column does not vanish).

2.3. Long time limit and maximal eigenvalue

In the long time limit, $t \to \infty$, the behaviour of $F_t(\mathcal{C})$ is dominated by the largest eigenvalue $\lambda(\gamma)$ of the matrix $M(\gamma)$:

$$F_t(\mathcal{C}) \to \mathrm{e}^{E_{\max}(\gamma) t} \langle \mathcal{C} | E_{\max}(\gamma) \rangle, \tag{6}$$

where the ket $|E_{\max}(\gamma)\rangle$ is the eigenvector corresponding to the largest eigenvalue. Therefore, when $t \to \infty$, we obtain

$$\langle \mathrm{e}^{\gamma Y_t} \rangle = \sum_{\mathcal{C}} F_t(\mathcal{C}) \sim \mathrm{e}^{E_{\max}(\gamma) t}. \tag{7}$$

More precisely, we have

$$\lim_{t\to\infty} \frac{1}{t} \log \langle \mathrm{e}^{\gamma Y_t} \rangle = E_{\max}(\gamma). \tag{8}$$

The function $E_{\max}(\gamma)$ contains the complete information about the cumulants of the total current Y_t in the long time limit. For example, the total current J and the diffusion constant Δ are given by

$$J = \lim_{t\to\infty} \frac{\langle Y_t \rangle}{t} = \left.\frac{\mathrm{d}E_{\max}(\gamma)}{\mathrm{d}\gamma}\right|_{\gamma=0}, \tag{9}$$

$$\Delta = \lim_{t\to\infty} \frac{\langle Y_t^2 \rangle - \langle Y_t \rangle^2}{t} = \left.\frac{\mathrm{d}^2 E_{\max}(\gamma)}{\mathrm{d}\gamma^2}\right|_{\gamma=0}. \tag{10}$$

Thus, the cumulants of Y_t can be determined by carrying out a perturbative expansion of $E_{\max}(\gamma)$ with respect to γ (a similar method has been used, in a different context, in [35]). The importance of the maximal eigenvalue $E_{\max}(\gamma)$ of the matrix $M(\gamma)$ also stems from the fact that it is closely related to the large deviation function G for the total current. We recall that the large deviation function G is defined as

$$G(j) = \lim_{t\to\infty} \frac{1}{t} \log\left[\text{Prob}\left(\frac{Y_t}{t} = j\right)\right]. \tag{11}$$

From equations (7) and (11), we find

$$\langle e^{\gamma Y_t}\rangle \sim e^{E_{\max}(\gamma)t} \sim \int e^{t(G(j)+\gamma j)}\, dj, \tag{12}$$

and deduce by the saddle-point approximation that the maximal eigenvalue $E_{\max}(\gamma)$ is the Legendre transform of the large deviation function $G(j)$

$$E_{\max}(\gamma) = \max_j (G(j) + \gamma j). \tag{13}$$

2.4. *Restatement of the problem*

We want to study the statistical properties of the total current in the partially asymmetric exclusion process. We have seen that in the long time limit, the maximal eigenvalue $E_{\max}(\gamma)$ of the deformed matrix $M(\gamma)$ is the generating function of the cumulants of the current, i.e. the power-series expansion of $E_{\max}(\gamma)$ in the vicinity of $\gamma = 0$ allows us to determine the statistical properties of the current. In the following sections, we shall first explain how to diagonalize the matrix $M(\gamma)$ using the Bethe ansatz; this method will allow us to write any eigenvalue of $M(\gamma)$ as a symmetric function of the roots of a system of coupled polynomial equations (the Bethe equations). Then, we shall develop a perturbative scheme to expand the maximal eigenvalue $E_{\max}(\gamma)$ in powers of γ, when $\gamma \to 0$. The first-order expansion will give us the current J and the second-order term will lead to the diffusion constant Δ.

3. The Bethe equations

The deformed matrix $M(\gamma)$ can be diagonalized by the Bethe ansatz. A vector P over the configuration space is an eigenvector of $M(\gamma)$ if it satisfies

$$M(\gamma)P = E(\gamma)P. \tag{14}$$

By representing a configuration by the positions of the N particles on the ring, $(r_1, r_2, \ldots, r_N)$ with $1 \leqslant r_1 < r_2 < \cdots < r_N \leqslant L$, the eigenvalue equation (14) becomes

$$\begin{aligned} EP(r_1,\ldots,r_N) = {} & \sum_i p[e^{\gamma} P(r_1,\ldots,r_{i-1}, r_i - 1, r_{i+1},\ldots,r_N) - P(r_1,\ldots,r_n)] \\ & + \sum_j q[e^{-\gamma} P(r_1,\ldots,r_{j-1}, r_j + 1, r_{j+1},\ldots,r_N) - P(r_1,\ldots,r_N)], \end{aligned} \tag{15}$$

where the sum runs over the indices i such that $r_{i-1} < r_i - 1$ and over the indices j such that $r_j + 1 < r_{j+1}$; these conditions ensure that the corresponding jumps are allowed. Following the coordinate *Bethe ansatz*, we assume that the eigenvector P can be written in the form

$$P(r_1,\ldots,r_n) = \sum_{\sigma\in\Sigma_n} A_\sigma z_{\sigma(1)}^{r_1} z_{\sigma(2)}^{r_2} \cdots z_{\sigma(n)}^{r_n}, \tag{16}$$

where Σ_n is the group of the $n!$ permutations of n indices. The coefficients $\{A_\sigma\}$ are rational functions of the fugacities $\{z_1,\ldots,z_n\}$. The expression (16) represents an eigenvector of $M(\gamma)$ if $\{z_1,\ldots,z_n\}$ satisfy the *Bethe equations* [21, 30]:

$$z_i^L = (-1)^{N-1} \prod_{j=1}^{N} \frac{q\, e^{-\gamma} z_i z_j - (p+q) z_i + p\, e^{\gamma}}{q\, e^{-\gamma} z_i z_j - (p+q) z_j + p\, e^{\gamma}} \qquad \text{for} \quad i = 1,\ldots,N, \tag{17}$$

and the corresponding eigenvalue of $M(\gamma)$ is given by

$$E(\gamma; z_1, z_2 \dots z_N) = p\,\mathrm{e}^{\gamma} \sum_{i=1}^{N} \frac{1}{z_i} + q\,\mathrm{e}^{-\gamma} \sum_{i=1}^{N} z_i - N(p+q). \tag{18}$$

For $\gamma = 0$, we know that the maximal eigenvalue of the Markov matrix M is equal to 0 and corresponds to the degenerate solution $z_i = 1$ for all i.

Remark. *The Gallavotti–Cohen Invariance.* The Bethe equations (17) and equation (18) are invariant under the transformation $z \to \frac{1}{z}, \gamma \to \log \frac{q}{p} - \gamma$. This symmetry implies that the spectrum of $M(\gamma)$ and that of $M\big(\log \frac{q}{p} - \gamma\big)$ are identical. This functional identity is satisfied in particular by the largest eigenvalue of M and we have $E_{\max}(\gamma) = E_{\max}\big(\log \frac{q}{p} - \gamma\big)$. This identity implies, using equation (13), that the large deviation function satisfies the symmetry

$$G(j) = G(-j) - \left(\log \frac{q}{p}\right) j. \tag{19}$$

This relation is a special case of the general *Fluctuation Theorem* valid for a large class of systems far from equilibrium. It was derived for more general Markovian systems in [12].

3.1. A useful change of variables

We introduce N auxiliary variables $(y_1, \dots, y_N)$ defined as

$$y_i = \frac{1 - \mathrm{e}^{-\gamma} z_i}{1 - x\,\mathrm{e}^{-\gamma} z_i} \qquad \text{for} \quad i = 1, \dots, N, \tag{20}$$

where we have introduced the asymmetry parameter x:

$$x = \frac{q}{p}. \tag{21}$$

We remark that the change of variables (20) is ill-defined for $x = 1$ which corresponds to the symmetric exclusion process. In the following, our calculations will always be performed for $x < 1$. Our results will extend to the symmetric case by taking the limit $x \to 1$ in the final expressions. The Bethe equations (17) now become

$$\mathrm{e}^{L\gamma} \left(\frac{1 - y_i}{1 - x y_i}\right)^{L} = -\prod_{j=1}^{N} \frac{y_i - x y_j}{x y_i - y_j} \qquad \text{for} \quad i = 1, \dots, N. \tag{22}$$

These equations are simpler than the original ones because they involve only linear polynomials in y_i's. By taking the product of the Bethe equations (22) over all the values of i, we obtain

$$\left(\mathrm{e}^{N\gamma} \prod_{i=1}^{N} \frac{1 - y_i}{1 - x y_i}\right)^{L} = (-1)^N \prod_{i,j=1}^{N} \frac{y_i - x y_j}{x y_i - y_j} = (-1)^{N+N^2} = 1. \tag{23}$$

This relation stems from the translation invariance of the model (momentum conservation).

In terms of y_i's, the eigenvalue (18) reads

$$E(\gamma) = p(1-x) \sum_{i=1}^{N} \left(\frac{1}{1 - y_i} - \frac{1}{1 - x y_i}\right). \tag{24}$$

When $\gamma \to 0$, all the roots $y_i(\gamma)$ that correspond to the maximal eigenvalue $E_{\max}(\gamma)$ of $M(\gamma)$ converge to the degenerate solution $\lim_{\gamma \to 0} y_i = 0$ and the maximal eigenvalue of $M(\gamma)$ also converges to 0. Using equation (23), we therefore find that, for small enough values of γ, the roots $y_i(\gamma)$ satisfy the relation

$$\mathrm{e}^{N\gamma}\prod_{i=1}^{N}\frac{1-y_i}{1-xy_i}=1. \tag{25}$$

This relation, which is a simple consequence of the Bethe equations, will be useful in the following to select the Bethe roots that correspond to $E_{\max}(\gamma)$.

3.2. The TASEP case

The Bethe equations (22) are a coupled nonlinear system of polynomial equations in the variables $y_1, \ldots, y_N$. Deriving exact results from these equations is a daunting task. However, for the special case of the totally asymmetric exclusion process (TASEP), which corresponds to $p = 1$ and $q = x = 0$, the Bethe equations can be reduced to an effective one-variable problem. Indeed, for $x = 0$, equations (22) read

$$\mathrm{e}^{L\gamma}(1-y_i)^L=(-1)^{N-1}\frac{y_i^N}{\prod_{j=1}^{N}y_j}. \tag{26}$$

Thus, all the Bethe roots y_i are solutions of the one-variable polynomial equation

$$\mathrm{e}^{L\gamma}(1-T)^L+CT^N=0, \tag{27}$$

where the constant C must be determined self-consistently by the relation

$$C=(-1)^N\prod_{j=1}^{N}\frac{1}{y_j}. \tag{28}$$

This crucial 'decoupling' property of the Bethe equations for $x = 0$, has lead to an exact calculation of the TASEP spectral gap [20, 21, 24] and has allowed Derrida and Lebowitz to calculate the complete large deviation function of the current for any finite values of L and N [31]. This effective decoupling also explains the spectral degeneracies of the TASEP Markov matrix [25]. Hence, the use of the Bethe ansatz has been restricted mostly to TASEP (for a review see, e.g. [30]).

For the partially asymmetric exclusion process, the Bethe equations are highly coupled to one another and cannot be simply reduced to an effective one-variable equation. Because of this technical difficulty for $0 < x < 1$, it has not been possible to extract from the Bethe ansatz any exact solution for finite systems. However, when $L \to \infty$, the Bethe equations reduce to an integro-differential equation for the density of roots, which was analyzed by Kim *et al* [22, 23] to derive the spectral gap and the current large deviation function.

4. Reformulation of the Bethe equations

We note that in the N Bethe equations (22) all the variables y_i play a similar role. This remark suggests that we should introduce an auxiliary variable T that plays a symmetric role with respect to all y_i's. We suppose that T satisfies the following equation:

$$\mathrm{e}^{L\gamma}\left(\frac{1-T}{1-xT}\right)^L=-\prod_{j=1}^{N}\frac{T-xy_j}{xT-y_j}\qquad\text{for}\quad i=1,\ldots,N, \tag{29}$$

where y_i's are now interpreted as *parameters* of the problem. This expression can be rewritten as a one-variable polynomial equation for the unknown T

$$P(T) = 0 \quad \text{with} \quad P(T) = \mathrm{e}^{L\gamma}(1-T)^L \prod_{j=1}^{N}(xT - y_j) + (1-xT)^L \prod_{j=1}^{N}(T - xy_j). \tag{30}$$

The N Bethe equations (22) imply that y_i is a root of $P(T)$ for $i = 1, \ldots, N$. Thanks to the auxiliary variable T, the Bethe equations have been reduced to an effective one- variable problem with N parameters. We can now proceed as follows: (i) find the roots of the polynomial $P(T)$ with the unknown T and with N parameters $y_1, \ldots, y_N$. (ii) Select N roots, amongst the $L+N$ solutions of $P(T) = 0$, and identify these selected roots to y_i's. This identification leads to N self-consistent equations (recall that for TASEP we had only one self-consistency condition).

It is possible to perform these steps using contour integration in the complex plane as in the TASEP case [24, 27, 31, 32]. However, the calculations will be greatly simplified if the problem is formulated in a purely algebraic manner, as follows. Let us define the polynomial $Q(T)$ as

$$Q(T) = \prod_{j=1}^{N}(T - y_j). \tag{31}$$

The roots of Q are exactly the Bethe roots $y_1, \ldots, y_N$ (equivalently, Q is the generating function of the symmetric polynomials in $y_1, \ldots, y_N$). The polynomial $P(T)$, defined in equation (30), can then be written as follows:

$$P(T) = \mathrm{e}^{L\gamma}(1-T)^L Q(xT) + (1-xT)^L x^N Q\left(\frac{T}{x}\right). \tag{32}$$

The fact that the Bethe roots $y_1, \ldots, y_N$ are roots of the polynomial $P(T)$ implies that $Q(T)$ *divides* $P(T)$. Therefore there exists a polynomial $R(T)$ of degree L such that $P(T) = Q(T)R(T)$, i.e. such that

$$Q(T)R(T) = \mathrm{e}^{L\gamma}(1-T)^L Q(xT) + (1-xT)^L x^N Q\left(\frac{T}{x}\right). \tag{33}$$

Substituting $T = y_i$ into this equation and taking into account that y_i is a root of $Q(T)$ we obtain

$$\mathrm{e}^{L\gamma}\left(\frac{1-y_i}{1-xy_i}\right)^L = -x^N \frac{Q\left(\frac{y_i}{x}\right)}{Q(xy_i)}. \tag{34}$$

Using expression (31) for $Q(T)$, we find that this relation is identical to the Bethe equation (22). We remark that this reformulation of the Bethe equations as a problem of polynomial divisibility has been used in various contexts [36–38] and is closely related to the functional Bethe ansatz [36, 38, 39].

4.1. Expression of the eigenvalue

The eigenvalue $E(\gamma)$, defined in equation (24), can be expressed in terms of the polynomial $Q(T)$:

$$E(\gamma) = p(1-x)\left(\frac{Q'(1)}{Q(1)} - \frac{1}{x}\frac{Q'(1/x)}{Q(1/x)}\right). \tag{35}$$

This formula can be simplified with the help of the 'QR-equation' (33) as follows. Substituting $T = 1$ into equation (33) we find

$$Q(1)R(1) = (1-x)^L x^N Q\left(\frac{1}{x}\right). \tag{36}$$

If we differentiate equation (33) with respect to T and then substitute $T = 1$ we obtain

$$Q'(1)R(1) + Q(1)R'(1) = -Lx^{N+1}(1-x)^{L-1}Q\left(\frac{1}{x}\right) + x^{N-1}(1-x)^L Q'\left(\frac{1}{x}\right). \tag{37}$$

Taking the ratio of the last two equations, we find that $E(\gamma)$ can be rewritten as

$$\frac{E(\gamma)}{p(1-x)} = -\frac{Lx}{1-x} - \frac{R'(1)}{R(1)}. \tag{38}$$

This is the expression of $E(\gamma)$ that will be used in the following.

Equation (25), that allows us to select the roots y_i corresponding to the maximal eigenvalue, is similarly rewritten in terms of Q and R as follows:

$$\mathrm{e}^{N\gamma}\frac{Q(1)}{x^N Q\left(\frac{1}{x}\right)} = 1. \tag{39}$$

Using equation (36), an alternative form is obtained

$$R(1) = \mathrm{e}^{N\gamma}(1-x)^L. \tag{40}$$

This relation will be very useful in the following work to simplify some calculations.

5. Perturbative solution of the functional Bethe ansatz equations

In this section, we explain how to solve equation (33) order by order in γ for the roots y_i that correspond to the maximal eigenvalue of the matrix $M(\gamma)$.

We first develop the polynomials Q and R in powers of γ

$$Q(T) = \prod_{j=1}^{N}(T-y_j) = \sum_{n=0}^{\infty}\gamma^n Q_n(T) = Q_0(T) + \gamma Q_1(T) + \gamma^2 Q_2(T) + \ldots \tag{41}$$

$$R(T) = \sum_{n=0}^{\infty}\gamma^n R_n(T) = R_0(T) + \gamma R_1(T) + \gamma^2 R_2(T) + \ldots. \tag{42}$$

We note that the degree of the polynomials $Q_n(T)$ for $n \geqslant 1$ is at most $N-1$. For $\gamma = 0$, we know that $E_0 = 0$ and that this maximal eigenvalue is obtained for $y_i = 0$. Therefore, we have

$$Q_0(T) = T^N \qquad \text{and} \qquad R_0(T) = (1-xT)^L + x^N(1-T)^L. \tag{43}$$

By substituting the power series (41) and (42) in the QR-equation (33), we obtain a hierarchical system of linear equations for the polynomials $R_n(T)$ and $Q_n(T)$. This system can be solved order by order by using the known 'initial conditions' $Q_0(T)$ and $R_0(T)$.

We now solve the QR-equation to the first and second orders.

5.1. First-order calculation

At first order, the QR-equation (33) becomes

$$Q_1(T)[(1-xT)^L + x^N(1-T)^L] + T^N R_1(T) = (1-T)^L Q_1(xT) + (1-xT)^L x^N Q_1\left(\frac{T}{x}\right) + Lx^N(1-T)^L T^N, \quad (44)$$

and the auxiliary equation (39) becomes

$$Q_1(1) - x^N Q_1\left(\frac{1}{x}\right) = -N. \quad (45)$$

It is simpler to define the polynomial

$$B_1(T) = Q_1(T) - x^N Q_1\left(\frac{T}{x}\right), \quad (46)$$

and to rewrite equations (44) and (45) as follows:

$$(1-xT)^L B_1(T) - (1-T)^L B_1(xT) = T^N(Lx^N(1-T)^L - R_1(T)). \quad (47)$$

$$B_1(1) = -N. \quad (48)$$

Because $B_1(T)$ and $Q_1(T)$ are of degree $\leqslant N-1$ and noting that the term on the rhs of equation (47) is divisible by T^N, we can reduce this equation modulo T^N and write

$$(1-xT)^L B_1(T) - (1-T)^L B_1(xT) \equiv 0 \quad [T^N]. \quad (49)$$

This equation allows us to determine the polynomial $B_1(T)$ up to a multiplicative constant β_0

$$B_1(T) \equiv \beta_0(1-T)^L \quad [T^N], \qquad \text{i.e.} \quad B_1(T) = \beta_0 \sum_{k=0}^{N-1}(-1)^k \binom{L}{k} T^k. \quad (50)$$

The constant β_0 is fixed using equation (48). Using the binomial identity (B.3), we find

$$-N = \beta_0 \sum_{k=0}^{N-1}(-1)^k\binom{L}{k} = \beta_0(-1)^{N-1}\binom{L-1}{N-1}, \qquad \text{i.e.} \quad \beta_0 = \frac{(-1)^N L}{\binom{L}{N}}. \quad (51)$$

From this relation it follows that

$$Q_1(T) = \sum_{k=0}^{N-1} q_k^{(1)} T^k \qquad \text{with} \qquad q_k^{(1)} = \frac{(-1)^{N+k}L}{\binom{L}{N}} \frac{\binom{L}{k}}{1-x^{N-k}}. \quad (52)$$

Using this formula and equation (47) the following exact expression for $R_1(T)$ is obtained:

$$R_1(T) = Lx^N(1-T)^L + (-1)^N \frac{L}{\binom{L}{N}} \sum_{p=0}^{N-1}\sum_{r=0}^{L}(-1)^{p+r}\binom{L}{p}\binom{L}{r}(x^p - x^r)T^{p+r-N}. \quad (53)$$

All negative powers of T in the above expression cancel out for the following reason: the coefficient of a term of the type T^{-d} with $d > 0$ is obtained by imposing the condition $p + r = N - d$ to the double sum in equation (53). Because of this condition, the indices p and r can vary only from 0 to $N - d$ and they both have the same effective range. The sum in equation (53) is antisymmetric with respect to p and r and therefore it vanishes. This proves that $R_1(T)$ is indeed a polynomial.

5.2. Second-order calculation

At second order, the polynomial $B_2(T)$ defined as

$$B_2(T) = Q_2(T) - x^N Q_2\left(\frac{T}{x}\right), \tag{54}$$

satisfies the following equation:

$$(1-xT)^L B_2(T) - (1-T)^L B_2(xT) = L(1-T)^L Q_1(xT) - R_1(T)Q_1(T) + T^N\left(x^N \frac{L^2}{2}(1-T)^L - R_2(T)\right). \tag{55}$$

If we write this relation modulo T^N we obtain the simpler equation

$$(1-xT)^L B_2(T) - (1-T)^L B_2(xT) \equiv L(1-T)^L Q_1(xT) - R_1(T)Q_1(T) \quad [T^N], \tag{56}$$

where the expressions for $Q_1(T)$ and $R_1(T)$ are given in equations (52) and (53), respectively. At order 2, the auxiliary equation (39) becomes

$$B_2(1) = Q_2(1) - x^N Q_2\left(\frac{1}{x}\right) = -NQ_1(1) - \frac{N^2}{2}. \tag{57}$$

The polynomial $B_2(T)$ is the sum of a special solution $\tilde{B}_2(T)$ of equation (56) and of a term that is proportional to $B_1(T)$, the solution of the homogeneous equation (49), i.e.

$$B_2(T) = \tilde{B}_2(T) + CB_1(T). \tag{58}$$

The proportionality constant C is fixed by using the auxiliary equation (57), which leads to

$$C = \frac{\tilde{B}_2(1)}{N} + Q_1(1) + \frac{N}{2}, \tag{59}$$

where we have used $B_1(1) = -N$ from equation (48).

A special solution to the polynomial equation (56) is given by

$$\tilde{B}_2(T) = \sum_{k=0}^{N-1}(1-x^{N-k})q_k^{(2)}T^k, \tag{60}$$

with

$$q_k^{(2)} = \frac{(-1)^{N+k+1}L^2}{\binom{L}{N}^2}\frac{1}{1-x^{N-k}}\left\{\sum_{r=1}^{N-1}\frac{\binom{L}{N+r}\binom{L}{k-r}x^r}{1-x^r} + \sum_{r=0}^{N-1}\frac{\binom{L}{N+r}\binom{L}{k-r}}{1-x^{N+r-k}}\right\}. \tag{61}$$

The main steps to derive equation (61) are given in appendix A. Finally, the polynomial $Q_2(T)$ is given by the linear combination

$$Q_2(T) = \sum_{k=0}^{N-1} q_k^{(2)}T^k + C\sum_{k=0}^{N-1} q_k^{(1)}T^k, \tag{62}$$

where the constant C is given in equation (59).

6. Exact formulae for the mean current and its fluctuations

Solving the QR-equation allows us to calculate the expansion of the largest eigenvalue $E_{\max}(\gamma)$, order by order, and to calculate the cumulants of the total current. The largest eigenvalue $E_{\max}(\gamma)$, can be expanded with respect to the parameter γ as follows

$$E_{\max}(\gamma) = p(1-x)\sum_{n=0}^{\infty}\gamma^n E_n. \tag{63}$$

Using equations (38), (40) and (42), the expansion of $E_{\max}(\gamma)$ is given by

$$\frac{E_{\max}(\gamma)}{p(1-x)} = -\frac{Lx}{1-x} - \frac{R_0'(1)}{(1-x)^L} + \gamma\left(\frac{NR_0'(1)}{(1-x)^L} - \frac{R_1'(1)}{(1-x)^L}\right) + \gamma^2\left(-\frac{N^2R_0'(1)}{2(1-x)^L} + \frac{NR_1'(1)}{(1-x)^L} - \frac{R_2'(1)}{(1-x)^L}\right) + \cdots \tag{64}$$

From expression (43) for $R_0(T)$, we find that

$$\frac{R_0'(1)}{(1-x)^L} = -\frac{Lx}{1-x}, \tag{65}$$

and we verify that the zeroth-order term E_0 in $E_{\max}(\gamma)$ vanishes.

6.1. Calculation of the current

The current J, defined in equation (9), corresponds to the coefficient of γ in the expansion of $E_{\max}(\gamma)$. To determine $R_1'(1)$, we start with equation (37) and expand it to the first order in γ:

$$\begin{aligned}\frac{R_1'(1)}{(1-x)^L} &= -N^2 + \frac{Lx}{1-x}\left(Q_1(1) - x^N Q_1\left(\frac{1}{x}\right)\right) - Q_1'(1) + x^{N-1}Q_1'\left(\frac{1}{x}\right)\\ &= -N^2 + \frac{Lx}{1-x}B_1(1) - B_1'(1),\end{aligned} \tag{66}$$

where in the last equality we have used the definition of $B_1(T)$ as given by equation (46). We know that $B_1(1) = -N$ from equation (48); the value of $B_1'(1)$ is readily obtained from the expression of $B_1(T)$ given in equations (50) and (51):

$$B_1'(1) = -LN\frac{N-1}{L-1}. \tag{67}$$

Thus, we have

$$\frac{R_1'(1)}{(1-x)^L} = -N\left(\frac{Lx}{1-x} + \frac{L-N}{L-1}\right). \tag{68}$$

Substituting this expression in the coefficient of γ in equation (64), we find that the total current is given by

$$J = p(1-x)\frac{N(L-N)}{L-1}. \tag{69}$$

This value agrees, of course, with the known formula, which is obtained very simply by using the fact that all the stationary configurations of ASEP on a ring are equiprobable. We recall that J represents the total current in the system; the current through a bond is given by J/L. Using the Bethe ansatz to find J is certainly a very complicated and distorted way to retrive a back-of-an-envelope calculation. However, J is one of the simplest quantity associated with ASEP and the fact that nobody could extract such an elementary formula from the Bethe equations has been a standing puzzle for a long time.

6.2. Calculation of the diffusion constant

The second-order term in the perturbative expansion (64) allows us to calculate the diffusion constant. Indeed, thanks to equation (10), we find that $\Delta = 2p(1-x)E_2$. Therefore, we have

$$\Delta = 2p(1-x)\left(-\frac{N^2R_0'(1)}{2(1-x)^L} + \frac{NR_1'(1)}{(1-x)^L} - \frac{R_2'(1)}{(1-x)^L}\right). \tag{70}$$

Hence, in order to calculate Δ, we also need $R_2'(1)$, which is determined in appendix B. After gathering all relevant terms, we are finally lead to the exact formula for the diffusion constant of the total current for the partially asymmetric exclusion process on a ring:

$$\Delta = \frac{2p(1-x)L}{(L-1)\binom{L}{N}^2} \sum_{r=1}^{N} r^2 \frac{1+x^r}{1-x^r} \binom{L}{N+r} \binom{L}{N-r}. \tag{71}$$

This formula agrees, of course, with the one obtained using the matrix representation method [34] (in that work, the fluctuations of the current through a bond were calculated exactly, i.e. Δ/L^2). From this exact expression, it is possible to deduce by finite size scaling that a tagged particle in an infinite system exhibits an anomalous diffusive behaviour with exponent $1/3$ (instead of one $1/2$). By taking the continuous limit $L \to \infty$ of equation (71) in the weakly asymmetric regime $x \to 1$, with scaling variable $\phi = (1-x)\sqrt{L}$, it is possible to derive a scaling function for the KPZ equation that describes the cross-over from the linear Edwards–Wilkinson regime to the nonlinear KPZ regime. We refer for more details to [34].

We emphasize that the calculation of Δ with the Bethe ansatz is of the same order of complexity as with the matrix method [34] but it is much simpler mathematically. The Bethe ansatz requires only elementary mathematical objects such as polynomials and involves systematic calculations, whereas for the matrix ansatz one has to find (guess) a suitable algebra, prove that this algebra solves the problem and then evaluate traces of various operators requiring the use of remarkable identities on q-binomials [34].

Furthermore, to calculate the higher cumulants of the current, one has to solve the QR-equation (33) to the suitable order in γ. By contrast, there is absolutely no clue on how to extend the matrix method to calculate, for example, the third cumulant of the current: the form of the algebra involved (if such an algebra does exist) is totally unknown.

7. Conclusion

Most of the analytical studies of the ASEP are based on two different techniques, the matrix product method and the Bethe ansatz. The matrix representation is suitable to calculate stationary state observables, such as correlations, phase diagrams etc. A major drawback of this method is that there is no constructive method to generate matrices that are suitable for a given stochastic model: one has to rely on educated guesses, after some trials and errors. Nevertheless, the matrix method, when applicable, is efficient and allows us to derive elegant combinatorial results for finite systems. On the contrary, the Bethe ansatz is a systematic procedure with such a wide range of applicability that it has grown into a subfield of theoretical physics: the theory of integrable systems. There exists *a priori* conditions, such as the Yang–Baxter relation, that insure that a system is integrable (i.e. it can be analyzed by the Bethe ansatz). Many methods have been developed to cope with the Bethe equations [36, 39]. However, it is very difficult to extract information for finite systems from the Bethe equations and usually one has to analyze these equations in the thermodynamic limit.

For the TASEP, the Bethe equations have a fundamental decoupling property that has lead to many exact results [21, 24, 25] and in particular to the calculation of an exact formula for the large deviation function [31, 32]. For the partially asymmetric case, the Bethe equations are strongly coupled and therefore they have been rarely used. The only exact results derived from them were obtained by Kim *et al* in the limit of an infinite size [22, 23]. In this paper, we have been able to overcome this technical difficulty thanks to a reformulation of the Bethe equations as a mere problem of polynomial divisibility that can be solved perturbatively in the fugacity parameter. We have calculated the mean value J of the current and its fluctuations

Δ. Obviously, the calculation of J from the Bethe ansatz is much more difficult than the elementary derivation. However, the calculation of Δ with the Bethe ansatz is less difficult than that with the matrix method [34]. Furthermore, the perturbative analysis of the Bethe ansatz can be extended *a priori* to any order to derive higher cumulants of the current. It is not known if the matrix method can be applied to such calculations.

The reformulation of the Bethe equations that we used here, is akin to the functional Bethe ansatz [36, 38, 39]. This method can be generalized to many other problems: higher moments of the ASEP current, subleading correction to the large deviation function of the symmetric exclusion process, systems with different classes of particles. We also believe that the method followed here could be applied to the ASEP with open boundaries for which the Bethe equations have been derived recently [40, 41]. For the open TASEP with all rates equal to one, it is known from the matrix method that the mean stationary current is given by the ratio of two consecutive Catalan numbers [13]: can this rather simple result be derived from the Bethe ansatz?

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Appendix A. Derivation of equation (61)

We want to derive formula (61) for $\tilde{B}_2(T)$ which is a particular solution of equation (56). We substitute the formal expression (60) of $\tilde{B}_2(T)$ into equation (56) and use the known explicit formulae for $Q_1(T)$ and $R_1(T)$ (given in equations (52) and (53), respectively). After identifying the terms of the same degree in T, the following linear system of equations is obtained:

$$\sum_{\substack{k+p=m\\0\leqslant k\leqslant N-1\\0\leqslant p\leqslant L}}\binom{L}{p}(x^p-x^k)c_k=\sum_{\substack{k+p+r=m+N\\0\leqslant k,p\leqslant N-1\\0\leqslant r\leqslant L}}\binom{L}{r}\binom{L}{p}\binom{L}{k}\frac{x^p-x^r}{1-x^{N-k}}-\binom{L}{N}\sum_{\substack{k+p=m\\0\leqslant k\leqslant N-1\\0\leqslant p\leqslant L}}\binom{L}{p}\binom{L}{k}x^k, \tag{A.1}$$

where we have introduced

$$c_k=(-1)^{N+k+1}\frac{\binom{L}{N}^2}{L^2}(1-x^{N-k})q_k^{(2)}. \tag{A.2}$$

The system (A.1) is a triangular system of N equations, parametrized by the integer m, with $0\leqslant m\leqslant N-1$. In equation (A.1), we have written explicitely the ranges for all the dummy variables. However, some pieces of information are redundant: for example, we know that $0\leqslant m\leqslant N-1$; therefore, if $k+p=m$ then both k and p must lie between 0 and $N-1$ (recall that a binomial coefficient with a negative entry is equal to 0). In the following, we shall not write such superfluous information.

We now start by transforming the rhs of equation (A.1): we note that the first sum on the rhs is formally antisymmetric with respect to the indices r and p. However, this sum does not vanish identically because the range of these two variables is not the same. If the range of r were from 0 to $N-1$, the total sum would be equal to zero. In other words, the terms in the range $0 \leqslant r \leqslant N-1$ do not contribute to the sum, only the terms with $N \leqslant r \leqslant L$ contribute. This sum is thus given by

$$\sum_{r=N}^{L} \sum_{\substack{0\leqslant k,p\leqslant N-1\\ k+p+r=m+N}} \binom{L}{r}\binom{L}{p}\binom{L}{k}\frac{x^p - x^r}{1-x^{N-k}} = \sum_{r=0}^{L-N}\binom{L}{N+r}\sum_{\substack{k+p+r=m\\ 0\leqslant k,p\leqslant N-1}}\binom{L}{p}\binom{L}{k}\frac{x^p - x^{N+r}}{1-x^{N-k}}$$
$$= \sum_{r=0}^{L-N}\binom{L}{N+r}\sum_{k+p=m}\binom{L}{p}\binom{L}{k-r}\left(x^k + \frac{x^p - x^k}{1-x^{N+r-k}}\right), \quad \text{(A.3)}$$

where, we have first replaced the dummy variable r by $r-N$ and then, to derive the last equality, we use the identity $(x^p - x^{N+r})/(1-x^{N-k}) = x^{k+r} + (x^p - x^{k+r})/(1-x^{N-k})$ and replace k by $k-r$. Thus, we rewrite the rhs of equation (A.1) as follows:

$$\sum_{r=0}^{L-N}\binom{L}{N+r}\sum_{k+p=m}\binom{L}{p}\binom{L}{k-r}\left(x^k + \frac{x^p - x^k}{1-x^{N+r-k}}\right) - \binom{L}{N}\sum_{k+p=m}\binom{L}{p}\binom{L}{k}x^k$$
$$= \sum_{r=1}^{L-N}\binom{L}{N+r}\sum_{k+p=m}\binom{L}{p}\binom{L}{k-r}x^k$$
$$+ \sum_{r=0}^{L-N}\binom{L}{N+r}\sum_{k+p=m}\binom{L}{p}\binom{L}{k-r}\frac{x^p - x^k}{1-x^{N+r-k}}. \quad \text{(A.4)}$$

The first term in the last equality is now rewritten using the following identity:

$$\sum_{k+p=m}\binom{L}{p}\binom{L}{k-r}x^k = \sum_{k+p=m}\binom{L}{p}\binom{L}{k-r}\frac{x^p - x^k}{1-x^r}x^r. \quad \text{(A.5)}$$

(This identity is readily proved after multiplying both sides by $(1-x^r)$, cancelling the common x^{k+r} term and noting that the remaining terms are identical up to a notation change.)

Finally, the initial system (A.1) becomes:

$$\sum_{k+p=m}\binom{L}{p}(x^p - x^k)c_k = \sum_{r=1}^{L-N}\binom{L}{N+r}\sum_{k+p=m}\binom{L}{p}\binom{L}{k-r}\frac{x^p - x^k}{1-x^r}x^r$$
$$+ \sum_{r=0}^{L-N}\binom{L}{N+r}\sum_{k+p=m}\binom{L}{p}\binom{L}{k-r}\frac{x^p - x^k}{1-x^{N+r-k}}. \quad \text{(A.6)}$$

Clearly, the solution of this equation is given by

$$c_k = \sum_{r=1}^{N-1}\frac{x^r\binom{L}{N+r}\binom{L}{k-r}}{1-x^r} + \sum_{r=0}^{N-1}\frac{\binom{L}{N+r}\binom{L}{k-r}}{1-x^{N+r-k}}. \quad \text{(A.7)}$$

This ends the proof of the formula (61).

Appendix B. Some useful steps in the calculation of Δ

B.1. Binomial formulae

In the following, we shall use repeatedly the following elementary binomial formulae:

$$p\binom{L}{p} = L\binom{L-1}{p-1}, \quad \text{(B.1)}$$

$$(L-p)\binom{L}{p} = L\binom{L-1}{p}. \tag{B.2}$$

$$\sum_{p=A}^{B}(-1)^p\binom{L}{p} = \sum_{p=A}^{B}(-1)^p\left\{\binom{L-1}{p}+\binom{L-1}{p-1}\right\}$$
$$= (-1)^B\binom{L-1}{B} + (-1)^A\binom{L-1}{A-1}, \tag{B.3}$$

$$\sum_{r=0}^{N} r\binom{L}{N+r}\binom{L}{N-r} = \frac{L}{2}\sum_{r=0}^{N}\left\{\binom{L-1}{N+r-1}\binom{L-1}{N-r} - \binom{L-1}{N+r}\binom{L-1}{N-r-1}\right\}$$
$$= \frac{L}{2}\binom{L-1}{N-1}\binom{L-1}{N} = \frac{N(L-N)}{2L}\binom{L}{N}^2. \tag{B.4}$$

B.2. An expression for the diffusion constant

We start with equation (37) and expand the polynomials Q and R to the second order in γ. This allows us to derive the following expression for $R_2'(1)$. We obtain

$$Q_0(1)R_2'(1) + Q_1(1)R_1'(1) + Q_2(1)R_0'(1) + Q_0'(1)R_2(1) + Q_1'(1)R_1(1) + Q_2'(1)R_0(1)$$
$$= -Lx^{N+1}(1-x)^{L-1}Q_2\left(\frac{1}{x}\right) + x^{N-1}(1-x)^L Q_2'\left(\frac{1}{x}\right). \tag{B.5}$$

We know from equation (43) that $Q_0(1) = 1, Q_0'(1) = N, R_0(1) = (1-x)^L$. From equation (40), we deduce $R_1(1) = N(1-x)^L$ and $R_2(1) = N^2(1-x)^L/2$. Finally, equations (65) and (68) give the values of $R_0'(1)$ and $R_1'(1)$. We also use equation (57) to express $Q_2(1/x)$ in terms of $Q_1(1)$ and $Q_2(1)$. Substituting this information into the previous expression leads to (remark that terms proportional to $Q_2(1)$ cancel out):

$$\frac{R_2'(1)}{(1-x)^L} = -\frac{N^3}{2} - \frac{N^2Lx}{2(1-x)} + N\frac{L-N}{L-1}Q_1(1) - NQ_1'(1) - Q_2'(1) + x^{N-1}Q_2'\left(\frac{1}{x}\right). \tag{B.6}$$

Inserting this expression into formula (70) for Δ gives

$$\frac{\Delta}{2p(1-x)} = \frac{N^3}{2} - N^2\frac{L-N}{L-1} - N\frac{L-N}{L-1}Q_1(1) + NQ_1'(1) + B_2'(1), \tag{B.7}$$

where we have used the definition (54) of the polynomial $B_2(T)$. With the help of equations (58), (59) and (67) we get

$$B_2'(1) = \tilde{B}_2'(1) + B_1'(1)\left(\frac{\tilde{B}_2(1)}{N} + Q_1(1) + \frac{N}{2}\right)$$
$$= \tilde{B}_2'(1) - L\frac{N-1}{L-1}\tilde{B}_2(1) - LN\frac{N-1}{L-1}Q_1(1) - LN^2\frac{N-1}{2(L-1)}. \tag{B.8}$$

Substituting this expression in equation (B.9), we conclude that

$$\frac{\Delta}{2p(1-x)} = -N^2\frac{L-N}{2(L-1)} - N^2Q_1(1) + NQ_1'(1) + \tilde{B}_2'(1) - L\frac{N-1}{L-1}\tilde{B}_2(1). \tag{B.9}$$

The values of all the terms that appear in this equation are known. We now evaluate each of these terms separately.

B.3. Calculation of some exact expressions

The value of $Q_1(1)$ is easily obtained from expression (52) of $Q_1(T)$

$$Q_1(1) = \frac{(-1)^N L}{\binom{L}{N}} \sum_{r=0}^{N-1} \frac{(-1)^r \binom{L}{r}}{1 - x^{N-r}} = \frac{L}{\binom{L}{N}} \sum_{r=1}^{N} \frac{(-1)^r \binom{L}{N-r}}{1 - x^r}. \tag{B.10}$$

Similarly, we have

$$Q_1'(1) = \frac{L}{\binom{L}{N}} \sum_{r=1}^{N} \frac{(-1)^r (N-r) \binom{L}{N-r}}{1 - x^r}. \tag{B.11}$$

To calculate $\tilde{B}_2(1)$, we start from the formula for $\tilde{B}_2(T)$ given in equations (60) and (61):

$$\tilde{B}_2(1) = \frac{(-1)^{N+1} L^2}{\binom{L}{N}^2} \sum_{k=0}^{N-1} (-1)^k \left\{ \sum_{r=1}^{N-1} \frac{x^r \binom{L}{N+r}\binom{L}{k-r}}{1 - x^r} + \sum_{r=0}^{N-1} \frac{\binom{L}{N+r}\binom{L}{k-r}}{1 - x^{N+r-k}} \right\}. \tag{B.12}$$

Exchanging the double sum and using equations (B.1) and (B.3), we rewrite the first term on the rhs of this expression as follows:

$$\sum_{r=1}^{N-1} \frac{x^r \binom{L}{N+r}}{1 - x^r} \sum_{k=0}^{N-1} (-1)^k \binom{L}{k-r} = \frac{(-1)^{N-1}}{L} \sum_{r=1}^{N-1} \frac{x^r (N-r) \binom{L}{N+r}\binom{L}{N-r}}{1 - x^r}. \tag{B.13}$$

In the second term on the rhs of (B.12), we remark that the effective range of the variable r is from 0 to k and we replace r by $r' = k - r$. We then transform this term in a manner similar to that described in equation (B.13). Finally, expression (B.12) simplifies to

$$\tilde{B}_2(1) = \frac{L}{\binom{L}{N}^2} \sum_{r=1}^{N} \frac{x^r (N-r) + N + r}{1 - x^r} \binom{L}{N+r}\binom{L}{N-r} - \frac{LN}{\binom{L}{N}} \sum_{r=1}^{N} \frac{(-1)^r \binom{L}{N-r}}{1 - x^r}. \tag{B.14}$$

Using similar steps, we find that $\tilde{B}_2'(1)$ is given by

$$\tilde{B}_2'(1) = \frac{L}{(L-1)\binom{L}{N}^2} \sum_{r=1}^{N} \frac{x^r (N-r)(LN - r - L) + (N+r)(LN + r - L)}{1 - x^r} \times \binom{L}{N+r}\binom{L}{N-r} - \frac{LN}{(L-1)\binom{L}{N}} \sum_{r=1}^{N} \frac{(-1)^r (LN - Lr + r - L)\binom{L}{N-r}}{1 - x^r}. \tag{B.15}$$

To conclude our calculation, we must substitute equations (B.10), (B.11), (B.14) and (B.15) into formula (B.9) for the diffusion constant. We find that all the terms that contain only one binomial factor, i.e. terms proportional to $(-1)^r \binom{L}{N-r} / (1 - x^r)$ cancel out amongst themselves. After some elementary simplifications, we are left with

$$\frac{\Delta}{p(1-x)} = \frac{2L}{(L-1)\binom{L}{N}^2} \sum_{r=1}^{N} r^2 \frac{1 + x^r}{1 - x^r} \binom{L}{N+r}\binom{L}{N-r} + \frac{2LN}{(L-1)\binom{L}{N}^2} \sum_{r=0}^{N} r \binom{L}{N+r}\binom{L}{N-r} - N^2 \frac{L-N}{(L-1)}. \tag{B.16}$$

The last two terms cancel with each other according to the identity (B.4). This ends the proof of equation (71).

Appendix C. Functional Bethe ansatz for TASEP

We consider here the special case of the TASEP (which corresponds to $p = 1$ and $q = x = 0$). We explain how to retrieve from the QR-equation (33) the parametric representation of $E_{\max}(\gamma)$ that was obtained in [31] by using contour integrals. For $x = 0$, the functional equation (33) reduces to

$$Q(T)R(T) = T^N + (-1)^{N-1}B(1-T)^L \qquad \text{with} \qquad B = (-1)^{N-1}\,\mathrm{e}^{L\gamma}\,Q(0). \tag{C.1}$$

From equation (43), we find that the zeroth-order polynomials for the TASEP are simply given by

$$Q_0(T) = T^N \qquad \text{and} \qquad R_0(T) = 1. \tag{C.2}$$

The perturbative expansions (41) and (42) can be rewritten as

$$Q(T) = T^N + \gamma\mathcal{Q}(T) \qquad \text{and} \qquad R(T) = 1 + \gamma\mathcal{R}(T), \tag{C.3}$$

where $\mathcal{Q}(T)$ is a polynomial of degree $N-1$ and $\mathcal{R}(T)$ is of degree $L-N$ (the coefficients of these two polynomials are functions of γ). We note, in particular, that $Q(0)$ is of order γ and thus B defined in equation (C.1) is also of order γ and is a small parameter. Dividing both sides of equation (C.1) by T^N and taking the logarithm, we obtain

$$\log\left(\frac{Q(T)}{T^N}\right) + \log R(T) = \log\left(1 + (-1)^{N-1}B\frac{(1-T)^L}{T^N}\right) = \sum_{k=1}^{\infty}\frac{(-1)^{Nk-1}B^k}{k}\frac{(1-T)^{kL}}{T^{kN}}, \tag{C.4}$$

where we have developed the logarithm in powers of B. We remark that the rhs of this equation is a series that contains both positive and negative powers of T. But, equation (C.3) implies that $Q(T)/T^N = 1 + \gamma\mathcal{Q}(T)/T^N$, i.e. $Q(T)/T^N$ is a polynomial in the variable $1/T$ of degree N. Therefore, the expansion of $\log(Q(T)/T^N)$ w.r.t. γ (or B) can only generate negative powers of T. Similarly, from equation (C.3) we have $\log R(T) = \log(1 + \gamma\mathcal{R}(T))$ and the expansion of this term can generate only positive powers of T. Therefore, the identification between the lhs and the rhs of equation (C.4) is unique and we have

$$\log\left(\frac{Q(T)}{T^N}\right) = \sum_{k=1}^{\infty}\frac{(-1)^{Nk-1}B^k}{k}\sum_{j=0}^{kN-1}(-1)^j\binom{kL}{j}T^{j-kN} \tag{C.5}$$

$$\log R(T) = \sum_{k=1}^{\infty}\frac{(-1)^{Nk-1}B^k}{k}\sum_{j=kN}^{kL}(-1)^j\binom{kL}{j}T^{j-kN}. \tag{C.6}$$

For the TASEP, equations (38) and (40) reduce to

$$E_{\max}(\gamma) = -\frac{R'(1)}{R(1)} = -\frac{\mathrm{d}}{\mathrm{dT}}\log R(T)\bigg|_{T=1} \qquad \text{and} \qquad \gamma = \frac{1}{N}\log R(1). \tag{C.7}$$

From equation (C.6), we obtain (with the help of equations (B.1) and (B.3) to calculate the sums over j)

$$E_{\max}(\gamma) = -N\sum_{k=1}^{\infty}B^k\frac{(kL-2)!}{(kN)!(kL-kN-1)!}, \tag{C.8}$$

$$\gamma = -\sum_{k=1}^{\infty}B^k\frac{(kL-1)!}{(kN)!(kL-kN)!}. \tag{C.9}$$

These two equations are precisely those derived in [31]. They provide a parametric formula for $E_{\max}(\gamma)$ that allows us to calculate the large deviation function of the current and its cumulants to any required order.

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